TRI-HARMONIC STRESS FUNCTION

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ABSTRACT. Airy's bi-harmonic Eqs. governs stress function in two dimensional isotropic elastic continumum. We find new Eqs. to determine stress functions in three dimensional isotropic elastic continumum. They are found to obey tri-harmonic Eqs.

VECTOR POTENTIAL TO STRESS TENSOR

Equilibrium of internal force in three dimensional elastic continumum reads

$$\partial_k f^{jk} = 0, \qquad \ldots \quad (1.1)$$

($j, k = 1, 2$ and 3; contracted over k),
with

$$f^{jk} = f^{kj},$$

where f^{jk} stands for stress tensor. We may take vector potential ϕ^1, ϕ^2 and ϕ^3 to them as

$$f^{jj} = -(\partial^2{}_k \phi^m + \partial^2{}_m \phi^k),$$

(j, k and $m = 1, 2$ and 3; cyclic),

$$f^{jk} = \partial_j \partial_k \phi^m (j \neq k \neq m), \qquad \ldots \quad (1.2)$$

so that (1.1) may identically be satisfied. We also take displacement potential by

$$e = (\partial_1 \psi_1, \partial_2 \psi_2, \partial_3 \psi_3,) \qquad \ldots \quad (1.3)$$

where

$$e = (e^1, e^2, e^3)$$

means displacement. Stress potential and displacement potential are related by

$$\phi^m(\vec{r}) = G[\psi^j(\vec{r}) + \psi^k(\vec{r})] + \chi(x^m), \qquad \ldots \quad (1.4)$$

with reference to the well known relations

$$f^{jk} = G(\partial_j e^k + \partial_k e^j). \qquad \ldots \quad (1.5)$$

$2G = mE/(m+1)$ stands for modulus of shearing elasticity, with

$$\vec{r} = (x^1, x^2, x^3).$$

E and m are modulus of longitudinal elasticity and Poisson's number, respectively. The function $\chi(x^m)$ depends only upon x^m, and vanishes if we take the boundary condition of

$$\psi^j(\pm\infty) = 0.$$
$$(j = 1, 2 \text{ and } 3).$$

TRI-HARMONIC STRESS FUNCTION

In the previous section, we found displacement potential to be related by the relation

$$2G\psi^j = \phi^k + \phi^m - \phi^j. \qquad \cdots \quad (2.1)$$

Substituting (2.1) into fundamental Eqs. of elasticity

$$\partial_j e^j. = (1/E)[\sigma^j - (\sigma^k + \sigma^m)/m],$$

one finds secular Eqs. of

$$\begin{pmatrix} \partial_1{}^2 + \Delta/m, & -(\partial_3{}^2 + \partial^2{}_1), & -(\partial_1{}^2 + \partial_2{}^2) \\ -(\partial_2{}^2 + \partial_3{}^2), & \partial_2{}^2 + \Delta/m, & -(\partial_1{}^2 + \partial_2{}^2) \\ -(\partial_2{}^2 + \partial_3{}^2), & -(\partial_3{}^2 + \partial_1{}^2), & \partial_3{}^2 + \Delta/m \end{pmatrix} \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix} = 0 \qquad \cdots \quad (2.2)$$

which leads us to

$$[(1+m)(1-m^2)/m^3]\Delta^3\phi^j = 0. \qquad \cdots \quad (2.3)$$

We finally obtain

$$\Delta^3\phi^j = 0, \qquad \cdots \quad (2.4)$$
$$(j = 1, 2 \text{ and } 3)$$

for

$$m \neq 1.$$

$$\sigma = (\sigma^1, \sigma^2, \sigma^3) = (f^{11}, f^{22}, f^{33})$$

means stress. Stress functions in three dimensional elastic continumum are governed by tri-harmonic Eqs.. There are six unknown functions f^{jk} and as many Eqs. of (1.1) and (2.4). f^{jk} can uniquely be determined by choosing three independent solutions of tri-harmonic Eqs. This is a generalization of Airy's bi-harmonic Eqs. into three dimensional isotropic elastic continumum.